

EXPLICIT FORMULAS FOR THE n -TH DERIVATIVES OF THE TANGENT AND COTANGENT FUNCTIONS

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ABSTRACT. In the paper, by induction, the author establishes explicit formulas for calculating the n -th derivatives of the tangent and cotangent functions.

1. MAIN RESULTS

It is common knowledge that the tangent function $\tan x$, defined by

$$\tan x = \frac{\sin x}{\cos x} \quad (1.1)$$

for $x \neq k\pi + \frac{\pi}{2}$ and $k \in \mathbb{Z}$, where \mathbb{Z} stands for the set of all integers, and the cotangent function $\cot x$, defined by

$$\cot x = \frac{\cos x}{\sin x} \quad (1.2)$$

for $x \neq k\pi$ and $k \in \mathbb{Z}$, are two of the simplest elementary functions.

It is well known that the sine function $\sin x$ and the cosine function $\cos x$ have the n -th derivatives

$$\sin^{(n)} x = \sin\left(x + \frac{n\pi}{2}\right) \quad \text{and} \quad \cos^{(n)} x = \cos\left(x + \frac{n\pi}{2}\right) \quad (1.3)$$

for $n \in \mathbb{N}$, where \mathbb{N} denotes the set of all positive integers. They are collected in a lot of textbooks for undergraduates and handbooks of mathematics. See, for example, [1, p. 77, 4.3.111 and 4.3.112]. However, the formulas for the n -th derivatives of the tangent function $\tan x$ and the cotangent function $\cot x$ can not be found anywhere, to the best of my ability.

The aim of this paper is to establish explicit formulas for the n -th derivatives of the tangent function $\tan x$ and the cotangent function $\cot x$.

Our main results may be stated as the following theorems.

Theorem 1.1. *For $n \in \mathbb{N}$, the derivatives of the tangent function may be computed by*

$$\tan^{(2n-1)} x = \frac{1}{\cos^{2n} x} \sum_{i=0}^{n-1} a_{2n-1,2i} \cos(2ix) \quad (1.4)$$

and

$$\tan^{(2n)} x = \frac{1}{\cos^{2n+1} x} \sum_{i=0}^{n-1} a_{2n,2i+1} \sin[(2i+1)x], \quad (1.5)$$

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where

$$a_{1,0} = 1, \quad (1.6)$$

$$a_{2n-1,0} = 2n \sum_{\ell=0}^{n-2} (-1)^\ell \binom{2n-1}{\ell} (n-\ell-1)^{2n-2} \quad (1.7)$$

for $n > 1$,

$$a_{2n-1,2i} = 2 \sum_{\ell=0}^{n-i-1} (-1)^{i+\ell} \binom{2n}{\ell} (n-i-\ell)^{2n-1} \quad (1.8)$$

for $1 \leq i \leq n-1$, and

$$a_{2n,2i+1} = 2 \sum_{\ell=0}^{n-i-1} (-1)^{i+\ell} \binom{2n+1}{\ell} (n-i-\ell)^{2n} \quad (1.9)$$

for $0 \leq i \leq n-1$.

Theorem 1.2. For $n \in \mathbb{N}$, the derivatives of the cotangent function may be calculated by

$$\cot^{(2n-1)} x = \frac{1}{\sin^{2n} x} \sum_{i=0}^{n-1} b_{2n-1,2i} \cos(2ix) \quad (1.10)$$

and

$$\cot^{(2n)} x = \frac{1}{\sin^{2n+1} x} \sum_{i=0}^{n-1} b_{2n,2i+1} \cos[(2i+1)x], \quad (1.11)$$

where

$$b_{2n-1,2i} = (-1)^{i+1} a_{2n-1,2i}, \quad (1.12)$$

$$b_{2n,2i+1} = (-1)^i a_{2n,2i+1}, \quad (1.13)$$

for $0 \leq i \leq n-1$ and $a_{p,q}$ for $p \in \mathbb{N}$ and $0 \leq q \leq p-1$ are defined by (1.6), (1.7), (1.8), and (1.9).

Remark 1.1. The equalities (1.8) and (1.9) can be unified as

$$a_{p,q} = (-1)^{\frac{1}{2}[p-\frac{3+(-1)^p}{2}]} 2 \sum_{\ell=0}^{\frac{p-q-1}{2}} (-1)^{\frac{p-q-1}{2}-\ell} \binom{p+1}{\ell} \left(\frac{p-q-1}{2} - \ell + 1 \right)^p \quad (1.14)$$

for $0 < q < p$.

Remark 1.2. By the way, the explicit formula for the n -th derivative of the exponential function $e^{1/x}$ was recently established in [2].

2. PROOFS OF THEOREMS

Now we are in a position to prove Theorems 1.1 and 1.2 by induction.

Proof of Theorem 1.1. We verify Theorem 1.1 by induction.

It is easy to obtain that

$$(\tan x)' = \sec^2 x = \frac{1}{\cos^2 x}$$

and

$$(\tan x)'' = 2 \tan x \sec^2 x = \frac{2 \sin x}{\cos^3 x}.$$

This means (1.6) and

$$a_{2,1} = 2. \quad (2.1)$$

Therefore, the formulas (1.4) and (1.5) are valid for $n = 1$.

Assume that the formulas (1.4) and (1.5) are valid for some $n > 1$. By this hypothesis and a direct differentiation, we have

$$\begin{aligned} \tan^{(2n+1)} x &= [\tan^{(2n)} x]' \\ &= \left\{ \frac{1}{\cos^{2n+1} x} \sum_{i=0}^{n-1} a_{2n,2i+1} \sin[(2i+1)x] \right\}' \\ &= \sum_{i=0}^{n-1} a_{2n,2i+1} \left\{ \frac{\sin[(2i+1)x]}{\cos^{2n+1} x} \right\}' \\ &= \frac{1}{\cos^{2(n+1)} x} \sum_{i=0}^{n-1} a_{2n,2i+1} \{ (n+1+i) \cos(2ix) + (i-n) \cos[2(i+1)x] \} \\ &= \frac{1}{\cos^{2(n+1)} x} \left\{ \sum_{i=1}^{n-1} [(n+1+i)a_{2n,2i+1} - (n+1-i)a_{2n,2i-1}] \cos(2ix) \right. \\ &\quad \left. + (n+1)a_{2n,1} - a_{2n,2n-1} \cos(2nx) \right\} \end{aligned}$$

and

$$\begin{aligned} \tan^{(2n+2)} x &= [\tan^{(2n+1)} x]' \\ &= \left[\frac{1}{\cos^{2n+2} x} \sum_{i=0}^n a_{2n+1,2i} \cos(2ix) \right]' \\ &= \sum_{i=0}^n a_{2n+1,2i} \left[\frac{\cos(2ix)}{\cos^{2n+2} x} \right]' \\ &= \frac{1}{\cos^{2n+3} x} \sum_{i=0}^n a_{2n+1,2i} \{ (n+1-i) \sin[(2i+1)x] - (n+1+i) \sin[(2i-1)x] \} \\ &= \frac{1}{\cos^{2n+3} x} \left\{ \sum_{i=1}^{n-1} [(n+1-i)a_{2n+1,2i} - (n+2+i)a_{2n+1,2(i+1)}] \sin[(2i+1)x] \right. \\ &\quad \left. + [2(n+1)a_{2n+1,0} - (n+2)a_{2n+1,2}] \sin x + a_{2n+1,2n} \sin[(2n+1)x] \right\}. \end{aligned}$$

Equating coefficients in

$$\begin{aligned} \frac{1}{\cos^{2(n+1)} x} \sum_{i=0}^n a_{2n+1,2i} \cos(2ix) &= \frac{1}{\cos^{2(n+1)} x} \left\{ (n+1)a_{2n,1} \right. \\ &\quad \left. - a_{2n,2n-1} \cos(2nx) + \sum_{i=1}^{n-1} [(n+1+i)a_{2n,2i+1} - (n+1-i)a_{2n,2i-1}] \cos(2ix) \right\} \end{aligned}$$

and

$$\frac{1}{\cos^{2n+3} x} \sum_{i=0}^n a_{2n+2,2i+1} \sin[(2i+1)x] = \frac{1}{\cos^{2n+3} x} \left\{ [2(n+1)a_{2n+1,0} - (n+2)a_{2n+1,2}] \sin x + a_{2n+1,2n} \sin[(2n+1)x] + \sum_{i=1}^{n-1} [(n+1-i)a_{2n+1,2i} - (n+2+i)a_{2n+1,2(i+1)}] \sin[(2i+1)x] \right\}$$

yields

$$a_{2n+1,0} = (n+1)a_{2n,1}, \quad (2.2)$$

$$a_{2n+1,2n} = -a_{2n,2n-1}, \quad (2.3)$$

$$a_{2n+2,1} = 2(n+1)a_{2n+1,0} - (n+2)a_{2n+1,2}, \quad (2.4)$$

$$a_{2n+2,2n+1} = a_{2n+1,2n}, \quad (2.5)$$

$$a_{2n+1,2i} = (n+1+i)a_{2n,2i+1} - (n+1-i)a_{2n,2i-1}, \quad (2.6)$$

$$a_{2n+2,2i+1} = (n+1-i)a_{2n+1,2i} - (n+2+i)a_{2n+1,2(i+1)}, \quad (2.7)$$

where $1 \leq i \leq n-1$.

In virtue of (2.3), (2.5), and (2.1), it is easy to obtain that

$$a_{2n,2n-1} = (-1)^{n-1}2 \quad (2.8)$$

and

$$a_{2n+1,2n} = (-1)^n 2. \quad (2.9)$$

Substituting (2.6) into the first term on the right hand side of (2.7) yields

$$a_{2n+2,2i+1} = [(n+1)^2 - i^2]a_{2n,2i+1} - (n+1-i)^2 a_{2n,2i-1} - (n+2+i)a_{2n+1,2i+2}. \quad (2.10)$$

Letting $i = n-1$ in (2.10) and making use of (2.8) and (2.9) give

$$a_{2n+2,2n-1} = (-1)^{n-1}2(1+6n) - 4a_{2n,2n-3}. \quad (2.11)$$

Substituting (2.2) into (2.4) shows

$$a_{2n+2,1} = 2(n+1)^2 a_{2n,1} - (n+2)a_{2n+1,2}. \quad (2.12)$$

Setting $n = 1$ in (2.12) and using (2.1) and (2.9) for $n = 1$ reveal

$$a_{4,1} = 8a_{2,1} - 3a_{3,2} = 22. \quad (2.13)$$

Combining (2.13) with (2.11), it is deduced that

$$a_{2n,2n-3} = (-1)^{n-1}2(2n+1-2^{2n}). \quad (2.14)$$

Substituting (2.8) and (2.14) into (2.6) for $i = n-1$ leads to

$$a_{2n+1,2n-2} = (-1)^n 2(2n+2-2^{2n+1}). \quad (2.15)$$

Taking $i = n-2$ in (2.10) and utilizing (2.14) and (2.15) give

$$\begin{aligned} a_{2n+2,2n-3} &= [(n+1)^2 - (n-2)^2]a_{2n,2n-3} - 9a_{2n,2n-5} - 2na_{2n+1,2n-2} \\ &= (-1)^n 2[3-4n-16n^2 + (10n-3)4^n] - 9a_{2n,2n-5}. \end{aligned} \quad (2.16)$$

Substituting (2.13) and (2.15) for $n = 2$ into (2.12) for $n = 2$ produces

$$a_{6,1} = 18a_{4,1} - 4a_{5,2} = 604. \quad (2.17)$$

Combination of (2.17) with the recursion (2.16) procures

$$a_{2n,2n-5} = (-1)^{n-1}2[n(2n+1) - (2n+1)2^{2n} + 3^{2n}]. \quad (2.18)$$

Substituting (2.14) and (2.18) into (2.6) for $i = n - 2$ leads to

$$a_{2n+1,2n-4} = (-1)^n 2[(n+1)(2n+1) - 2(n+1)2^{2n+1} + 3^{2n+1}]. \quad (2.19)$$

Taking $i = n - 3$ in (2.10) and utilizing (2.18) and (2.19) yield

$$\begin{aligned} a_{2n+2,2n-5} &= 8(n-1)a_{2n,2n-5} + (1-2n)a_{2n+1,2n-4} - 16a_{2n,2n-7} \\ &= (-1)^{n-1} 2[(2n+1)(10n^2 - 7n - 1) + (3+n-6n^2)2^{2n+2} \\ &\quad + (14n-11)9^n] - 16a_{2n,2n-7}. \end{aligned} \quad (2.20)$$

Substituting (2.17) and (2.19) for $n = 3$ into (2.12) for $n = 3$ produces

$$a_{8,1} = 32a_{6,1} - 5a_{7,2} = 31238. \quad (2.21)$$

Combination of (2.21) with the recursion (2.20) procures

$$a_{2n,2n-7} = (-1)^{n-1} 2 \left[\frac{n(2n-1)(2n+1)}{3} - n(2n+1)2^{2n} + (2n+1)3^{2n} - 4^{2n} \right]. \quad (2.22)$$

Substituting (2.18) and (2.22) into (2.6) for $i = n - 3$ leads to

$$\begin{aligned} a_{2n+1,2n-6} &= (-1)^n 2 \left[\frac{2n(n+1)(2n+1)}{3} - (n+1)(2n+1)2^{2n+1} \right. \\ &\quad \left. + 2(n+1)3^{2n+1} - 4^{2n+1} \right]. \end{aligned} \quad (2.23)$$

Letting $i = n - 4$ in (2.10) and utilizing (2.22) and (2.23) acquire

$$\begin{aligned} a_{2n+2,2n-7} &= 5(2n-3)a_{2n,2n-7} - 2(n-1)a_{2n+1,2n-6} - 25a_{2n,2n-9} \\ &= (-1)^{n-1} \frac{2}{3} [48n^4 - 28(3 \times 4^n + 2)n^3 + 6(2^{2n+3} + 16 \times 9^n - 3)n^2 \\ &\quad - (27 \times 2^{4n+1} + 20 \times 3^{2n+1} - 69 \times 4^n - 11)n + 69 \times 16^n \\ &\quad + 3 \times 4^{n+1} - 9^{n+2}] - 25a_{2n,2n-9}. \end{aligned} \quad (2.24)$$

Substituting (2.21) and (2.23) for $n = 4$ into (2.12) for $n = 4$ produces

$$a_{10,1} = 50a_{8,1} - 6a_{9,2} = 2620708. \quad (2.25)$$

Combination of (2.25) with the recursion (2.24) procures

$$\begin{aligned} a_{2n,2n-9} &= (-1)^{n-1} 2 \left[\frac{(n-1)n(2n-1)(2n+1)}{6} - \frac{n(2n-1)(2n+1)}{3} 2^{2n} \right. \\ &\quad \left. + n(2n+1)3^{2n} - (2n+1)4^{2n} + 5^{2n} \right]. \end{aligned} \quad (2.26)$$

Substituting (2.22) and (2.26) into (2.6) for $i = n - 4$ leads to

$$\begin{aligned} a_{2n+1,2n-8} &= (-1)^n 2 \left[\frac{n(n+1)(2n-1)(2n+1)}{6} - \frac{2n(n+1)(2n+1)}{3} 2^{2n+1} \right. \\ &\quad \left. + (n+1)(2n+1)3^{2n+1} - 2(n+1)4^{2n+1} + 5^{2n+1} \right]. \end{aligned} \quad (2.27)$$

From (2.8), (2.14), (2.18), (2.22), and (2.26), it is observed that the formulas (2.26) and (2.22) may be rearranged respectively as

$$a_{2n,2n-9} = (-1)^{n-1} 2 \left[\frac{(2n-2)(2n-1)(2n)(2n+1)}{4!} - \frac{(2n-1)(2n)(2n+1)}{3!} 2^{2n} \right]$$

$$\begin{aligned}
& + \frac{(2n)(2n+1)}{2!} 3^{2n} - \frac{2n+1}{1!} 4^{2n} + 5^{2n} \Big] \\
& = (-1)^{n-1} 2 \sum_{\ell=0}^4 (-1)^\ell \binom{2n+1}{\ell} (5-\ell)^{2n}
\end{aligned}$$

and

$$a_{2n, 2n-7} = (-1)^{n-1} 2 \sum_{\ell=0}^3 (-1)^{3-\ell} \binom{2n+1}{\ell} (4-\ell)^{2n}.$$

This reminds us to claim

$$a_{2n, 2n-2i-1} = (-1)^{n-1} 2 \sum_{\ell=0}^i (-1)^{i-\ell} \binom{2n+1}{\ell} (i-\ell+1)^{2n} \quad (2.28)$$

for $0 \leq i \leq n-1$. This can be verified without much difficulty by induction and along the lines used in the deduction of (2.8), (2.14), (2.18), (2.22), and (2.26). Furthermore, the formula (2.28) may be rewritten as more useful form

$$\begin{aligned}
a_{2n, 2i-1} &= (-1)^{n-1} 2 \sum_{\ell=0}^{n-i} (-1)^{n-i-\ell} \binom{2n+1}{\ell} (n-i-\ell+1)^{2n} \\
&= 2 \sum_{\ell=0}^{n-i} (-1)^{i+\ell+1} \binom{2n+1}{\ell} (n-i-\ell+1)^{2n}
\end{aligned} \quad (2.29)$$

for $1 \leq i \leq n$. The formula (1.9) is proved.

By virtue of (2.6) and (2.29), it follows that

$$\begin{aligned}
a_{2n+1, 2i} &= 2(n+1+i) \sum_{\ell=0}^{n-i-1} (-1)^{i+\ell} \binom{2n+1}{\ell} (n-i-\ell)^{2n} \\
&\quad - 2(n+1-i) \sum_{\ell=0}^{n-i} (-1)^{i+\ell+1} \binom{2n+1}{\ell} (n-i-\ell+1)^{2n} \\
&= 2 \sum_{\ell=0}^{n-i} (-1)^{i+\ell} \binom{2n+2}{\ell} (n-i-\ell+1)^{2n+1}
\end{aligned} \quad (2.30)$$

for $1 \leq i \leq n$. This can also be simply derived from

$$a_{2n+1, 2n-2i} = (-1)^n 2 \sum_{\ell=0}^i (-1)^{i-\ell} \binom{2n+2}{\ell} (i-\ell+1)^{2n+1} \quad (2.31)$$

for $0 \leq i \leq n-1$, which is inductively concluded from (2.9), (2.15), (2.19), (2.23), and (2.27). The formula (1.8) is proved.

From (2.2) and (2.29) applied to $i=1$, it is easy to obtain

$$a_{2n+1, 0} = 2(n+1) \sum_{\ell=0}^{n-1} (-1)^\ell \binom{2n+1}{\ell} (n-\ell)^{2n} \quad (2.32)$$

for $1 \leq i \leq n$. The formula (1.7) is proved. Theorem 1.1 is thus proved. \square

Proof of Theorem 1.2. This theorem can be proved by induction as the proof of Theorem 1.1. However, we would like to derive it from Theorem 1.2 as follows.

Since

$$\cot x = -\tan\left(x + \frac{\pi}{2}\right) \quad (2.33)$$

for $x \neq k\pi$ and $k \in \mathbb{Z}$, we have

$$\cot^{(n)} x = -\tan^{(n)}\left(x + \frac{\pi}{2}\right)$$

for $n \in \mathbb{N}$. By Theorem 1.1, we obtain

$$\begin{aligned} \cot^{(2n-1)} x &= -\frac{1}{\cos^{2n}\left(x + \frac{\pi}{2}\right)} \sum_{i=0}^{n-1} a_{2n-1,2i} \cos(2ix + i\pi) \\ &= \frac{1}{\sin^{2n} x} \sum_{i=0}^{n-1} (-1)^{i+1} a_{2n-1,2i} \cos(2ix) \end{aligned}$$

and

$$\begin{aligned} \cot^{(2n)} x &= -\frac{1}{\cos^{2n+1}\left(x + \frac{\pi}{2}\right)} \sum_{i=0}^{n-1} a_{2n,2i+1} \sin\left[(2i+1)\left(x + \frac{\pi}{2}\right)\right] \\ &= \frac{1}{\sin^{2n+1} x} \sum_{i=0}^{n-1} (-1)^i a_{2n,2i+1} \cos[(2i+1)x]. \end{aligned}$$

Thus, Theorem 1.2 is proved. \square

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